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A novel washout effect in the flavored leptogenesis

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Abstract: We investigate a flavored washout effect due to the decay of the lightest right-handed neutrino, assuming that there is non-vanishing initial lepton asymmetry and the decay of the lightest right-handed neutrinos gives negligible contribution to the asymmetry. We figure out general features of the washout effect. It is shown that there is a novel parameter region where an effect that is negligible in most cases plays a critical role and a sizable lepton asymmetry can survive against the washout process even in a strong washout region.

Keywords: Baryogenesis, Neutrino Physics, Beyond Standard Model, CP violation.

Contents

1. Introduction

Leptogenesis [1] is a simple mechanism to generate baryon number asymmetry of the Universe. The idea is that a lepton asymmetry produced at a high temperature is converted to the baryon asymmetry through the sphaleron interactions [2] which conserve $B-L$ number but break $B+L$ number. A simplest version of the leptogenesis is based on the seesaw mechanism [3] which can also explain the smallness of the neutrino masses by introducing heavy right-handed neutrinos (RHN) to the standard model. In the seesaw models, the CP-violating and out-of-equilibrium decay of the RHN can produce the lepton asymmetry.

The seesaw mechanism is also easy to be implemented in supersymmetric and/or grand unified theories (GUTs) which are most attractive candidates for the physics beyond the standard model. In such a class of model, the successful leptogenesis is considered as a mechanism to generate the baryon asymmetry of the Universe. However, in many models especially in GUT models, the mass (M_1) of the lightest RHN (N_1) is too small to generate the observed baryon asymmetry by the N_1 decay and the asymmetry should be produced through another mechanism.

Recently it was pointed out that flavor effects give significant contributions to the leptogenesis [4 − 10]. One of the interesting phenomena in the flavored leptogenesis is that the primordial lepton asymmetry generated by the decay of the second lightest RHN (N_2), of the inflaton or so can remain against the washout by the lightest one [6, 9]. This is an interesting possibility to give enough baryon asymmetry even when the mass of the lightest RHN is too small. In such a scenario, the study of the washout effect by the lightest RHN is very important.

In this paper, we study the detail of this flavored washout effect due to the lightest RHN and we point that there is a novel parameter region ((2b) in section 2.3) where an effect that is negligible in most cases plays an important role. This effect is due to off-diagonal elements of the so-called A-matrix, and thus unique in the flavored leptogenesis. Most recently, importance of the effect of the off-diagonal elements was first studied numerically in the context that N_1 decay generates the lepton asymmetry [10], while these effects were incorporated in the numerical calculations [8]. The authors concluded that the final (total) asymmetry was corrected only by a few percent due to the inclusion of the off-diagonal entries. Here, we adapt the analysis on the washout effect to the case where the asymmetry produced by the N_2 decay dominates the baryon asymmetry of the Universe, and show that a sizable lepton asymmetry can remain against the washout process in a different way from those studied in refs. [6, 9] (which correspond to (2a) and (3) with $M_1 \gg 10^9$GeV, respectively). Interestingly, in this case, the final (total) asymmetry can be enhanced by orders of magnitude compared to the result of neglecting the off-diagonal elements, in contrast to the case of the N_1 decay dominance [10].

In the section 2, we investigate the washout effect by the N_1 decay, assuming that there is primordial lepton asymmetry before the decay becomes relevant. In particular we study the behaviour of the solutions by perturbation with respect to the off-diagonal elements of A-matrix, given that these elements are small. In the section 3, we will show examples that the primordial asymmetry is generated by the decay of the second lightest RHN. The section 4 is devoted to summary and discussion.

2. Flavor dependence of the Washout effect

2.1 The seesaw mechanism and the leptogenesis

In the seesaw mechanism, RHN are introduced to the standard model,

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \sum_{f,i} Y_{fi}\bar{h}\bar{l}_f N_i - \sum_i \frac{M_i}{2}\bar{N}_i N_i^c + \text{h.c.} \quad (i = 1,2,3 \text{ and } f = e,\mu,\tau)\ , \tag{2.1}$$

with N_i, l_f, and h being RHN, lepton doublets, and Higgs doublet respectively. Here we take the basis where the Yukawa matrix for charged leptons and the mass matrix of RHN are diagonalized. Integrating out the heavy RHN and giving the VEV to Higgs, one can obtain the neutrino masses, m_i, and Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix [11], U, as

$$U^* \text{diag}(m_1, m_2, m_3) U^\dagger = v^2 Y \text{diag}(M_1^{-1}, M_2^{-1}, M_3^{-1}) Y^T\ . \tag{2.2}$$

where $v = 174$GeV is the Higgs VEV. Supposing the reheating temperature after the inflation is enough high, the RHN are produced through the interaction with the doublet leptons and the Higgs fields. When the temperature decreases down to the mass of RHN, the production becomes inefficient and RHN decay away. This out-of-equilibrium decay of the RHN generates $B-L$ asymmetry which is proportional to the CP violation in the decay, ϵ, defined as

$$\epsilon_i^f = \frac{\Gamma_{N_i \to l_f h} - \Gamma_{N_i \to \bar{l}_f \bar{h}}}{\sum_f \left(\Gamma_{N_i \to l_f h} + \Gamma_{N_i \to \bar{l}_f \bar{h}}\right)}. \tag{2.3}$$

This asymmetry is converted to the baryon asymmetry through the electroweak sphaleron [2] processes.

2.2 Boltzmann equation

In order to evaluate the baryon asymmetry of the Universe, the Boltzmann equation is used. In this analysis, for simplicity, we omit the scattering effects which are considered to be subdominant. The decays and inverse decays, $N_1 \leftrightarrow l_f h, \bar{l}_f \bar{h}$, are considered with rate γ_D^f. With this simplification, the evolution of the asymmetry of $\Delta_f = B/3 - L_f$ after the decoupling of the second lightest RHN is described by the following set of Boltzmann equations [8]:

$$\frac{dY_{N_1}}{dz} = -\frac{z}{sH(M_1)}\gamma_D\left(\frac{Y_{N_1}}{Y_{N_1}^{\rm eq}} - 1\right), \tag{2.4}$$

$$\frac{dy_{\Delta_f}}{dz} = -\frac{z}{sH(M_1)}\left[\gamma_D\epsilon_1^f\left(\frac{Y_{N_1}}{Y_{N_1}^{\rm eq}} - 1\right) + \frac{\gamma_D^f}{2}\left(\frac{y_{l_f}}{Y_{l_f}^{\rm eq}} + \frac{y_h}{Y_h^{\rm eq}}\right)\right], \tag{2.5}$$

where $z = M_1/T$ and $\gamma_D = \sum_f \gamma_D^f$. The parameters Y_X and $Y_X^{\rm eq}$ indicate the number density of the particle X divided by the entropy density $s = 2\pi^2 g_*^{\rm eff} T^3/45$ and its value in equilibrium respectively, and $y_X = Y_X - Y_{\bar{X}}$. The parameter $g_*^{\rm eff} \sim g_{\rm SM}^{\rm eff} = 106.75$ is the total effective number of the degrees of freedom (DOF) at the temperature around M_1. With these definitions, we have

$$Y_{N_1}^{\rm eq} = \frac{3}{4}\frac{45\zeta(3)g_{N_1}}{4\pi^4 g_*^{\rm eff}}z^2 K_2(z), \quad \frac{Y_{N_1}^{\rm eq}}{Y_{\rm massless}^{\rm eq}} = \frac{1}{2}\frac{g_{N_1}}{g_{\rm massless}}z^2 K_2(z)\begin{cases} 1 & \text{for fermion} \\ 3/4 & \text{for boson}\end{cases} \tag{2.6}$$

where $\zeta(x)$ is the Riemann's zeta function and $K_\nu(x)$ is the modified Bessel function. Here, g_X is (not effective) number of DOF of the particle X, for example $g_{l_f} = g_{\bar{l}_f} = 2$ and $g_{N_1} = 2$.

After neglecting the finite temperature effects such as the thermal masses and running of the couplings (for these effects, see ref. [14]) for simplicity, one can obtain γ_D and the Hubble parameter $H(z)$ as

$$\gamma_D = sY_{N_1}^{\rm eq}\frac{K_1(z)}{K_2(z)}\Gamma_D, \quad H(T) = \sqrt{\frac{8\pi^3 g_*^{\rm eff}}{90}}\frac{T^2}{M_{pl}}, \tag{2.7}$$

where $\Gamma_D = (Y^\dagger Y)_{11}M_1/(8\pi)$ is the total decay width of N_1 and $M_{pl} = 1.22 \times 10^{19}$GeV is the Planck scale. Now, let us define the "washout mass parameter" $\tilde{m}_i^f$ and "equilibrium neutrino mass parameter" m_* as

$$\tilde{m}_i^f = \frac{|Y_{fi}|^2 v^2}{M_i}, \quad m_* = \frac{H(M_1)\tilde{m}_1}{\Gamma_D} = \sqrt{\frac{8\pi^3 g_*^{\rm eff}}{90}}\frac{8\pi v^2}{M_{pl}} = 1.07\,\text{meV}, \tag{2.8}$$

where $\tilde{m}_i = \sum_f \tilde{m}_i^f$. The partial decay width to a flavor f is written as $\Gamma_D^f = \tilde{m}_1^f M_1^2/(8\pi v^2)$ and the total decay width is given by the sum of them $\Gamma_D = \sum \Gamma_D^f$. Eventually, the

Boltzmann equations are written as

$$\frac{dY_{N_1}}{dz} = -z\frac{K_1(z)}{K_2(z)}\frac{\tilde{m}_1}{m_*}\left(Y_{N_1} - Y_{N_1}^{\rm eq}\right), \tag{2.9}$$

$$\frac{dy_{\Delta_f}}{dz} = -z\frac{K_1(z)}{K_2(z)}\left[\epsilon_1^f\frac{\tilde{m}_1}{m_*}\left(Y_{N_1} - Y_{N_1}^{\rm eq}\right) + \frac{1}{4}\frac{\tilde{m}_1^f}{m_*}z^2K_2(z)\left(2\frac{y_{l_f}}{g_{l_f}} + \frac{3}{2}\frac{y_h}{g_h}\right)\right]. \tag{2.10}$$

The coefficient of y_h is different from that in ref. [8] by a factor 3/4 which comes from the relative factor of the number density in the equilibrium of fermion/boson (2.6). Notice that, however, in the derivation of the Boltzmann equations (2.4) and (2.5), an approximation $f = 1/\left(\exp\left((E-\mu)/T\right) \pm 1\right) \sim \exp\left(-(E-\mu)/T\right)$ is made. Within this approximation, the relative factor 3/4 (and the factor 2 in eq. (2.15)) disappears. In fact the right hand side of eq. (2.5) is originally written in terms of not $y_X/Y_X^{\rm eq}$ but the chemical potentials of the particle X, μ_X. In literatures, these chemical potentials are replaced as eq. (2.5) using the relation between μ_X and $y_X/Y_X^{\rm eq}$ with the approximation. If one would not use this replacement, a factor 1/2 appears instead of 3/4. This difference of the factor does not affect the results a lot in many cases and the term of y_h itself is often neglected. In our analysis, the contribution can affect the result significantly. In the following we take basically the factor 3/4 for illustration. It is straightforward to make analyses with a different factor.

An important point is that y_{Δ_f} is invariant under the standard evolution of the Universe after N_1 is decoupled and related to the present baryon asymmetry as $y_B = 12/37 \times \sum y_{\Delta_f}$ [15][1] at the weak scale due to the electroweak sphaleron process. Thus, we define "baryon asymmetry" by multiplying the factor 12/37 on y_{Δ_f} even at a higher temperature. Because this value is proportional to the present baryon to photon ratio as $\eta_B = g_0^{\rm eff}\pi^4/(45\zeta(3)) \times y_B = 7.04 y_B$ successful leptogenesis scenario should predict the "baryon asymmetry" $y_B^{\rm obs} = 0.87 \pm 0.03 \times 10^{-10}$ which comes from the observable $\eta_B^{\rm obs} = 6.1 \pm 0.2 \times 10^{-10}$ [17].

As mentioned in the introduction, in some models, the CP violation ϵ_1^f is too small to produce enough lepton asymmetry. In this case we can neglect the source term. Then, the evolution of y_{Δ_f} is controlled by one equation as

$$\frac{dy_{\Delta_f}}{dz} = -\frac{1}{4}z^3K_1(z)\frac{\tilde{m}_1^f}{m_*}\left(2\frac{y_{l_f}}{g_{l_f}} + \frac{3}{2}\frac{y_h}{g_h}\right) \tag{2.11}$$

Note that we assume that the asymmetries of the lepton doublet, y_{l_f}, and of the Higgs fields, y_h, are much smaller than 1 because it is proportional to the $B-L$ asymmetry as shown below, and thus neglect the higher terms. These relations are forced by the fast (spectator) processes, such as the sphaleron process, and depend on the temperature. For example at a high temperature, only the interactions mediated by the gauge and the top Yukawa coupling are in the thermal equilibrium, while at a lower temperature weaker interactions come in it. For instance, let us concentrate on the range of the temperature

[1]The factor 12/37 can be somewhat different, for instance 28/79 [16], depending on the timing of the freeze out of the electroweak sphaleron, but in any case, the value is approximately equal to 1/3.

where the interactions mediated by all the second and third generational Yukawa couplings are in the equilibrium but the first generational ones are not. This range is likely the one in which we are interested, namely $T \sim M_1 < 10^9$GeV. In this range, the weak sphaleron and the QCD sphaleron [18] are considered to occur fast enough.

These fast interactions make the following relations hold among the chemical potentials:

$$\begin{array}{ll}
\mu_{q_i} - \mu_{u_j} + \mu_h = 0, & i = 1,2,3 \text{ (due to the CKM mixing)} \\
\mu_{q_i} - \mu_{d_j} - \mu_h = 0, & j = 2,3 \\
\mu_{l_j} - \mu_{e_j} - \mu_h = 0, & \\
\sum_i \left(3\mu_{q_i} + \mu_{l_i}\right) = 0, & \text{EW sphaleron} \\
\sum_i \left(2\mu_{q_i} - \mu_{u_i} - \mu_{d_i}\right) = 0, & \text{QCD sphaleron.}
\end{array}$$

In addition to these relations, we impose the charge neutrality of the Universe and assume the vanishing asymmetries for the right handed leptons (and quarks if the QCD sphaleron is not considered) of the first generation:

$$\sum_i \left(\mu_{q_i} + 2\mu_{u_i} - \mu_{d_i} - \mu_{l_i} - \mu_{e_j}\right) + 2\mu_h = 0, \tag{2.12}$$

$$\mu_{e_1} = 0, \tag{2.13}$$

$$\mu_{u_1} = \mu_{d_1} (= 0 \quad \text{if no QCD sphaleron}) \tag{2.14}$$

Notice that in the eq. (2.12), the factor 2 in front of μ_h comes from the relative factor in the relation between the asymmetry density and the chemical potential for massless particles:

$$y_X = \frac{g_X \mu_X}{3s} T^2 \begin{cases} 1/2 & \text{for fermion} \\ 1 & \text{for boson} \end{cases} . \tag{2.15}$$

Taking care of this factor 2, we get similar relations among the asymmetries by replacing $\mu_{\text{fermion}} \to y_{\text{fermion}}/g_{\text{fermion}}$, $\mu_h \to 2y_h/g_h$ and $\sum_i \left(\frac{1}{3} \times 3 \times (2\mu_{q_i} + \mu_{u_i} + \mu_{d_i})\right)/3 - \left(2\mu_{l_f} + \mu_{e_f}\right) \to y_{\Delta_f}$. By solving these relations, we find the expression of y_{l_f}/g_{l_f} and y_h/g_h in terms of y_{Δ_i} as

$$\frac{y_{l_f}}{g_{l_f}} = \sum_{f'} C_{l\,ff'} y_{\Delta_{f'}}, \qquad \frac{3}{4}\frac{y_h}{g_h} = \sum_{f'} \frac{3}{4} C_{h\,f'} y_{\Delta_{f'}}, \tag{2.16}$$

with

$$C_l = \begin{pmatrix} -109/253 & 25/506 & 25/506 \\ 29/1012 & -493/1518 & 13/1518 \\ 29/1012 & 13/1518 & -493/1518 \end{pmatrix}, \qquad C_h = \begin{pmatrix} -53/506 \\ -37/253 \\ -37/253 \end{pmatrix} \tag{2.17}$$

if we do not consider the QCD sphaleron and with

$$C_l = \begin{pmatrix} -151/358 & 10/179 & 10/179 \\ 25/716 & -172/537 & 7/537 \\ 25/716 & 7/537 & -172/537 \end{pmatrix}, \qquad C_h = \begin{pmatrix} -37/358 \\ -26/179 \\ -26/179 \end{pmatrix} \tag{2.18}$$

if we take into account it. In the following, we examine only the latter case because there are no qualitative difference.

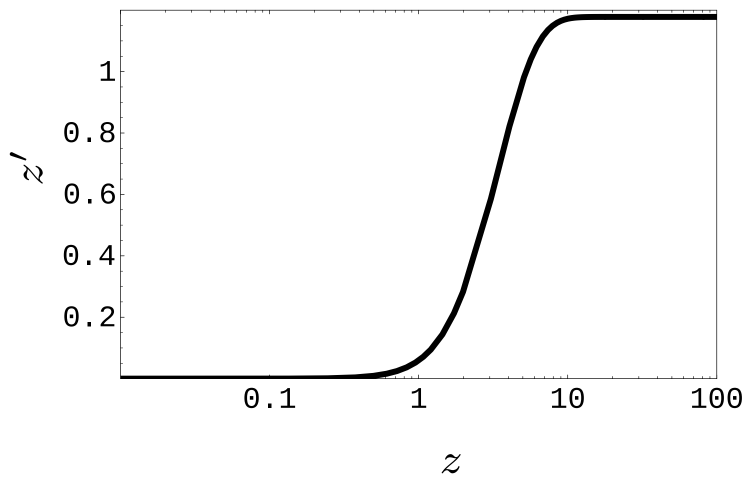

Figure 1: The new variable z' as a function of z.

From these expressions, we have the following Boltzmann equation:

$$\frac{dy_{\Delta_f}}{dz} = -\sum_{f'} \frac{z^3}{4} K_1(z) \frac{\tilde{m}_1^f}{m_*} A_{ff'} y_{\Delta_{f'}}, \tag{2.19}$$

with

$$A_{ff'} = \begin{pmatrix} 715/716 & 19/179 & 19/179 \\ 61/716 & 461/537 & 103/537 \\ 61/716 & 103/537 & 461/527 \end{pmatrix} = \begin{pmatrix} 1.00 & 0.11 & 0.11 \\ 0.085 & 0.86 & 0.19 \\ 0.085 & 0.19 & 0.86 \end{pmatrix}. \tag{2.20}$$

In order to analyse this equation, it is convenient to change the variable from z to z' that satisfy $dz'/dz = z^3 K_1(z)/4$ so that

$$\frac{\partial y_{\Delta_f}}{\partial z'} = -\sum_{f'} \frac{\tilde{m}_1^f}{m_*} A_{ff'} y_{\Delta_{f'}}. \tag{2.21}$$

The range of z' is from $z'(z=0) = 0$ to $z'_\infty = z'(z=\infty) = 3\pi/8 = 1.18$. The relation between them is shown in the figure 1.

This figure shows the washout occurs mostly in the temperature range $M_1/10 \lesssim T \lesssim 3M_1$.

For comparison, the Boltzmann equation for the usual one-flavor approximation, which is in reality valid only when the temperature is high enough that even the processes mediated by the tau Yukawa coupling are out-of-equilibrium, is given as[2]

$$\frac{\partial y_\Delta}{\partial z'} = -\frac{\tilde{m}_1}{m_*} y_\Delta, \tag{2.22}$$

where $y_\Delta = \sum y_{\Delta_f}$ and $\tilde{m}_1 = \sum \tilde{m}_1^f$.

[2]Here we neglect the Higgs contribution in order to compare our analysis with those in literatures, though it is considered in the next section.

2.3 Solutions

Roughly speaking, the matrix A is close to a diagonal one. And thus, we can find an approximate solution by a perturbation with respect to rather small off-diagonal elements. Namely,

$$y_{\Delta_f} = y^{(0)}_{\Delta_f} + y^{(1)}_{\Delta_f}\varepsilon + \mathcal{O}(\varepsilon^2)\ , \tag{2.23}$$

setting $A_{ff'} = \varepsilon \tilde{A}_{ff'}$ for $f \neq f'$ with $\varepsilon \ll 1$ and $\tilde{A}_{ff'} = \mathcal{O}(1)$.

Neglecting the off-diagonal elements, each initial asymmetry $y^0_{\Delta_f}$ is exponentially washed out. The evolution of the asymmetry is given by

$$y^{(0)}_{\Delta_f}(z') = \exp\left(-\frac{\tilde{m}^f_1}{m_*}A_{ff}z'\right) y^0_{\Delta_f}. \tag{2.24}$$

In many cases, the next leading order (NLO) gives only small contributions to total asymmetry (see (1) and (2a) in figure 2). If $y^{(0)}_{\Delta_f}$ is strongly suppressed as in the case of (2b) in figure 2, NLO contribution becomes significant.

When the off-diagonal elements are switched on, the asymmetry follows

$$\frac{\partial y^{(1)}_{\Delta_f}\varepsilon}{\partial z'} = -\frac{\tilde{m}^f_1}{m_*}A_{ff}y^{(1)}_{\Delta_f}\varepsilon - \sum_{f'\neq f}\frac{\tilde{m}^f_1}{m_*}A_{ff'}y^{(0)}_{\Delta_{f'}}. \tag{2.25}$$

Inserting the leading order solution eq. (2.24), we find

$$y^{(1)}_{\Delta_f}(z')\varepsilon = \sum_{f'\neq f}\frac{\frac{\tilde{m}^f_1}{m_*}A_{ff'}}{\frac{\tilde{m}^{f'}_1}{m_*}A_{f'f'} - \frac{\tilde{m}^f_1}{m_*}A_{ff}}\left(\exp\left(-\frac{\tilde{m}^{f'}_1}{m_*}A_{f'f'}z'\right) - \exp\left(-\frac{\tilde{m}^f_1}{m_*}A_{ff}z'\right)\right) y^0_{\Delta_{f'}}. \tag{2.26}$$

This expression shows that even if the initial asymmetry of a certain flavor is zero, the asymmetry is generated from those of the others although it is suppressed by a small off-diagonal element of A-matrix. For most cases, this order already gives good approximation for the final value of the total asymmetry. In fact this perturbation $y^{(0)}_{\Delta_f} + y^{(1)}_{\Delta_f}\varepsilon$ coincides with the numerical solution of the Boltzmann equation, eq. (2.21) within 10% (see the dashed line and solid line in figure 2, respectively).

(1) $\tilde{m}^f_1 \lesssim m_*$ $(f = e, \mu, \tau)$

In this case, one may expect the washout effect is small and thus the flavor effect can not play an important role. However, even in this case, the final total asymmetry can be 2 times larger than the one-flavor approximation (See (1) in figure 2).

(2) $\tilde{m}^{f_1}_1 \lesssim m_*$ and $\tilde{m}^{f_2}_1 \gtrsim m_*$ $(f_1 \neq f_2)$

In this case, the summation $\tilde{m}_1 = \sum \tilde{m}^f_1$ is dominated by $\tilde{m}^{f_2}_1$ and is larger than m_*. It is called strong washout region. As shown below, however, if we take account of the flavor effect the washout effect is drastically changed. The effect depends strongly on the flavor structure of the initial asymmetry.

In the following, let us take $\tilde{m}^e_1 \lesssim m_* \lesssim \tilde{m}^a_1$ $(a = \mu, \tau)$ as a representative example for clarity. It is straightforward to apply this analysis to the other cases. We consider two typical sets of the initial asymmetries:

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(2a) $y^0_{\Delta_e} \gtrsim y^0_{\Delta_a}$

In this case, the washout effect of y_{Δ_e} is controlled by $\tilde{m}^e_1$ which is much smaller than $\tilde{m}_1$, while y_{Δ_a} are generated due to the small off-diagonal elements of A_{ae}, with the opposite sign. Because of the large $\tilde{m}^a_1$, these generated y_{Δ_a} are washed out strongly and can not become comparable with y_{Δ_e}. Thus, in terms of the perturbation, the leading order approximation is sufficient.

Note that even in this case, the washout factor $y_\Delta(z'_\infty)/y^0_\Delta \sim \exp\left(-\tilde{m}^e_1 A_{ee}/m_*\right)$ is quite different (much larger) than the one-flavor approximation, $\exp\left(-\tilde{m}_1/m_*\right)$ (See (2a) in figure 2). This is the case even when $\tilde{m}^e_1$ is not so much smaller than the others, due to the exponential washout factor [6].

(2b) $y^0_{\Delta_e} \ll y^0_{\Delta_\mu}$ and/or y_{Δ_τ}

The asymmetry y_{Δ_a} decreases rapidly, while the asymmetry y_{Δ_e} produced due to the off-diagonal elements is washed out much more slowly. This means that at some point y_{Δ_e} becomes dominant. Once it becomes dominant, the following evolution is similar to the one in the case (2a). Thus, the washout factor is controlled basically by the small $\tilde{m}^e_1$ rather than $\tilde{m}_1$ or $\tilde{m}^a_1$ (See (2b) in figure 2). Interestingly in this case, the sign of the total $B-L$ asymmetry changes through the washout.

In the case of (2b) in figure 2, the difference between the total $B-L$ asymmetry with and without off-diagonal entries of A-matrix is found as large as four orders of magnitude. It means that the effect of off-diagonal elements of A-matrix gives really significant contributions.

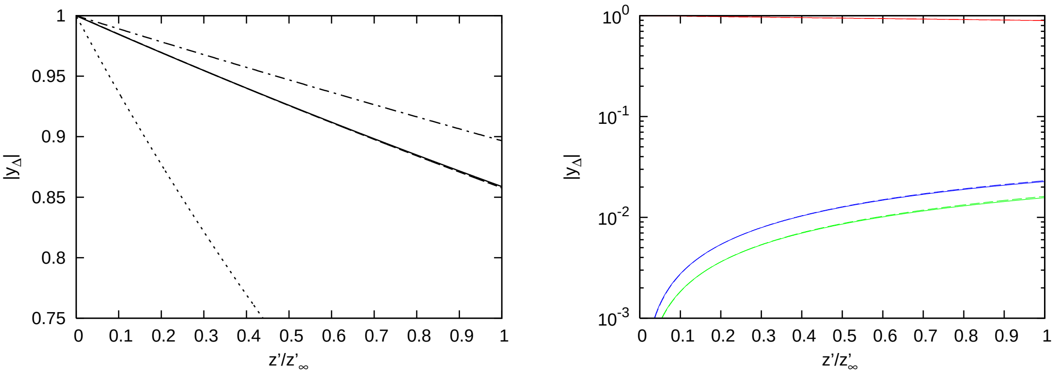
Note that the approximation at the NLO is quite bad for y_{Δ_a} because the secondary conversion from y_{Δ_e}, which is generated by the NNLO effect, is important. Nevertheless, the approximation for the total asymmetry is rather good because these are small as in the case (2a).

(3) $\tilde{m}^f_1 \gtrsim m_*$ $(f = e, \mu, \tau)$

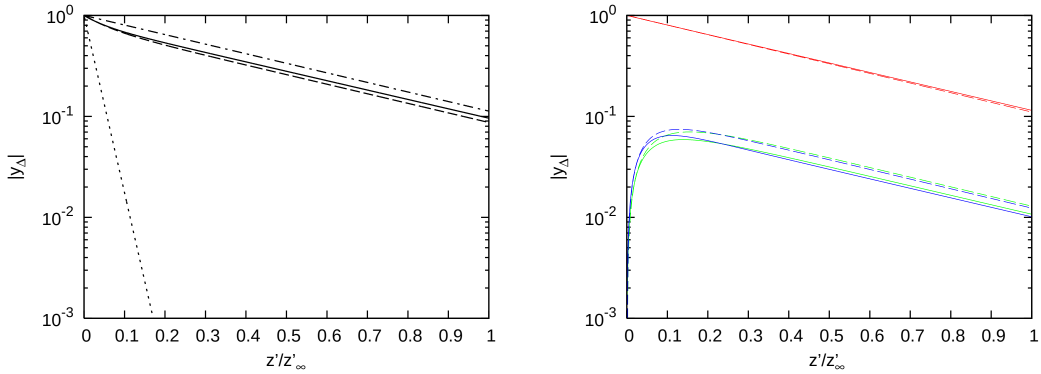
In this case, all the asymmetries in each flavors are strongly washed out. Thus, it is hard that the observed value remains after the washout, as far as $M_1 < 10^9$GeV.[3]

From the above considerations, it is clear that the washout factor is basically controlled by the smallest washout mass parameter. This is in great contrast to the non-flavored case, where the factor is controlled basically by the largest washout mass parameter. Interestingly, this is also true even for the case that the initial asymmetry of the flavor with smallest washout mass parameter is tiny (the case (2b)). For this case, the effect of the off-diagonal elements, which is usually negligible, is crucial.

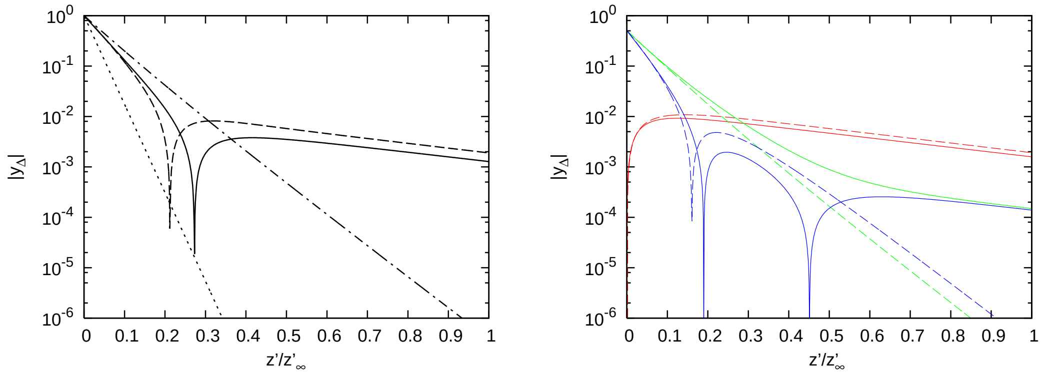
[3]If we consider models with $M_1 \gg 10^9$GeV where the muon Yukawa interaction is out-of-equilibrium, the asymmetry along with the direction in the flavor space that is orthogonal both to the direction determined by N_1 Yukawa coupling and τ direction is free from the washout, even in this case [9].



(1) $\{\tilde{m}_1^e, \tilde{m}_1^\mu, \tilde{m}_1^\tau\} = \{0.1, 0.2, 0.3\}\,\mathrm{meV}$, $\{y^0_{\Delta_e}, y^0_{\Delta_\mu}, y^0_{\Delta_\tau}\} = \{1, 0, 0\}$



(2a) $\{\tilde{m}_1^e, \tilde{m}_1^\mu, \tilde{m}_1^\tau\} = \{2, 15, 20\}\,\mathrm{meV}$, $\{y^0_{\Delta_e}, y^0_{\Delta_\mu}, y^0_{\Delta_\tau}\} = \{1, 0, 0\}$



(2b) $\{\tilde{m}_1^e, \tilde{m}_1^\mu, \tilde{m}_1^\tau\} = \{2, 15, 20\}\,\mathrm{meV}$, $\{y^0_{\Delta_e}, y^0_{\Delta_\mu}, y^0_{\Delta_\tau}\} = \{0, 0.5, 0.5\}$

Figure 2: The evolutions of the $B/3 - L_f$ asymmetries. The horizontal line is z'/z'_∞, and the vertical line is $|y_\Delta|$. In the left figures, solid, broken, dotted and dot-dashed lines show the total $B - L$ asymmetry calculated by solving the full Boltzmann equation (2.21), by the approximation formula $y^{(0)}_{\Delta_f} + y^{(1)}_{\Delta_f}\varepsilon$ (eqs. (2.24) and (2.26)), by the one-flavour approximation eq. (2.22) and by the Boltzmann equation (2.21) without the off-diagonal elements of A-matrix (corresponding to $y^{(0)}_{\Delta_f}$), respectively. In the right figures, red, green and blue lines respectively show the y_{Δ_e}, y_{Δ_μ} and y_{Δ_τ}, and the solid and broken lines correspond to the full solution of eq. (2.21) and eqs. (2.24) and (2.26), respectively.

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2.4 Fixed point

From eq. (2.21), one can obtain a coupled equations for $y_{\Delta_f}/y_{\Delta_\tau}$ ($f = e, \mu$) as

$$\frac{d}{dz'}\left(\frac{y_{\Delta_f}}{y_{\Delta_\tau}}\right) = \sum_{f'=e,\mu} \frac{\tilde{m}_1^\tau}{m_*}\left[\frac{\tilde{m}_1^f}{\tilde{m}_1^\tau} A_{ff'} - A_{\tau f}\left(\frac{y_{\Delta_f}}{y_{\Delta_\tau}}\right)\right]\left(\frac{y_{\Delta_{f'}}}{y_{\Delta_\tau}}\right) . \tag{2.27}$$

This couple of equations have fixed points in the space of $(y_{\Delta_e}/y_{\Delta_\tau}, y_{\Delta_\mu}/y_{\Delta_\tau})$, which is determined by solving the equations

$$\sum_{f'=e,\mu,\tau}\left[\frac{\tilde{m}_1^f}{\tilde{m}_1^\tau} A_{ff'} - A_{\tau f'}\left(\frac{y_{\Delta_f}}{y_{\Delta_\tau}}\right)\right]\left(\frac{y_{\Delta_{f'}}}{y_{\Delta_\tau}}\right) = 0 . \tag{2.28}$$

Once the flow of the solution reaches close to a fixed point at $z' = z'_{\rm fp}$, the set of ratios $(y_{\Delta_e}/y_{\Delta_\tau}, y_{\Delta_\mu}/y_{\Delta_\tau})$ becomes invariant and the Boltzmann equations can be rewritten as

$$\frac{dy_{\Delta_f}}{dz'} = -\frac{\tilde{m}_1^{\rm fp}}{m_*} y_{\Delta_f} , \quad z' \geq z'_{\rm fp} . \tag{2.29}$$

This means that the asymmetries of the all flavor are washed out with the universal washout mass parameter $\tilde{m}_1^{\rm fp}$ which corresponds to one of the eigenvalues of matrix $B_{ff'} \equiv \tilde{m}_1^f A_{ff'}$. There are three possible points in the case of three effective flavor numbers. However two of them are unstable fixed points and only one point is attractive. The attractive one most likely corresponds to the smallest eigenvalue which is smaller than the smallest $\tilde{m}_1^f$. Around the attractive fixed point, the asymmetry in the flavor with the smallest washout mass parameter, $y_{\Delta_{f_1}}$ dominates the total asymmetry. This can be understood as follows. The asymmetry in the flavor with larger washout mass parameter, $y_{\Delta_{f_2}}$ decrease more quickly as discussed in the case (2b) in the section 2.3. When $y_{\Delta_{f_2}}$ becomes the order of $A_{f_2 f_1} y_{\Delta_{f_1}}$, $y_{\Delta_{f_2}}$ evolves similar to $y_{\Delta_{f_1}}$ because the transportation from $y_{\Delta_{f_1}}$ becomes to control the evolution. Then the washout of $y_{\Delta_{f_1}}$ becomes weaker due to the transportation from $y_{\Delta_{f_2}}$.

When the initial condition is too far from the fixed point and $\tilde{m}_1 \ll m_*$, the washout term decouples from the system before the solution flows into the fixed point.[4]

One can see similar phenomena also in the case where only N_1 decay produces the asymmetries and they are washed out by the N_1 (inverse) decay, although, in this case, it seems difficult to get the large enhancement shown in the case (2b).

In fact the effect of off-diagonal elements of A-matrix in such a case was discussed in ref. [10] and the authors reached the conclusion that the total asymmetry got modified by only a few percent due to the effect, except for the case where the individual contributions cancel out each other to make such a tiny modification important. It was claimed there, however, that the individual asymmetry, y_{Δ_f}, especially the sub-dominant ones, were sensitive to the effect. This is understood as the conversions from the dominant one, even though suppressed by the small off-diagonal entries, are able to be much larger than the generations by the sub-dominant ones itself.

[4]Notice that the z' takes the value in the range of $0 \leq z' \leq 3\pi/8$ corresponding to $0 \leq z \leq \infty$.

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3. Asymmetries by the second lightest RHN decay

In this section, we investigate the possibility that the initial asymmetries are generated via the second lightest RHN decay.

For simplicity, we restrict ourselves to the case that the mass of the second lightest RHN is larger than 10^{12}GeV. This case is qualitatively discussed in ref. [9]. For this mass range, the fast interactions that are in the equilibrium when the RHN decays are only the interactions mediated by the gauge and the top Yukawa coupling and the QCD sphalerons. This means that the fast interactions can not distinguish all the generations of the lepton doublets, and thus two liner combinations of the three doublets that do not interact with the RHN are never produced. Namely, only $l_{\parallel} \propto Y_{\tau 2} l_\tau + Y_{\mu 2} l_\mu + Y_{e2} l_e$ are produced. Then, the relevant Boltzmann equations are for one flavor system, which is given by eqs. (2.9) and (2.10) by replacing all the index 1 to 2 (including $z \to M_2/T$) and suppressing the flavor indexes. With the definitions given in the section 2.2, we have

$$\frac{dY_{N_2}}{dz} = -\frac{\tilde{m}_2}{m_*} z \frac{K_1(z)}{K_2(z)} \left(Y_{N_2} - Y_{N_2}^{\rm eq} \right) \tag{3.1}$$

$$\frac{dy_\Delta}{dz} = -\epsilon_2 \frac{\tilde{m}_2}{m_*} z \frac{K_1(z)}{K_2(z)} \left(Y_{N_2} - Y_{N_2}^{\rm eq} \right) - \frac{z^3}{4} K_1(z) \frac{\tilde{m}_2}{m_*} A y_\Delta . \tag{3.2}$$

The A-factor for this case is calculated in a similar way to the discussion in the section 2.2 as

$$A = 2C_l + \frac{3}{2} C_h = \frac{67}{46} \tag{3.3}$$

It is possible, of course, that we solve these set of equations numerically to evaluate the $B - L$ asymmetry produced by the decay of the second lightest RHN. In this article, however, we use the following approximation formula proposed in ref. [19], which includes the effects of the scatterings, to evaluate the “baryon asymmetry”:

$$y_B \sim -\frac{12}{37 g_*^{\rm eff}} \epsilon_2 \eta \left(A \tilde{m}_2 \right) \tag{3.4}$$

with

$$\eta(x) = \left(\left(\frac{x}{8.25\,\text{meV}} \right)^{-1} + \left(\frac{0.2\,\text{meV}}{x} \right)^{-1.16} \right)^{-1} . \tag{3.5}$$

In any case, these equations are controlled by the parameters $\tilde{m}_2$ and ϵ_2 which are determined by the mass spectrum of the RHN M_i and the neutrino Yukawa coupling Y_{fi}. To be more concrete, they are respectively given by sums of (2.8) and

$$\epsilon_2^f = \frac{1}{8\pi} \frac{1}{(Y^\dagger Y)_{22}} \text{Im} \sum_{i \neq 2} Y_{f2}^* Y_{fi} \left(\left(Y^\dagger Y \right)_{2i} f\left(\frac{M_i^2}{M_2^2} \right) + \left(Y^\dagger Y \right)_{i2} g\left(\frac{M_i^2}{M_2^2} \right) \right) . \tag{3.6}$$

Here both the diagrams of the vertex correction and of the self-energy correction are implemented in each function as

$$f(x) = -\frac{\sqrt{x}}{x-1} + \sqrt{x} \left(1 - (1+x) \ln \left(\frac{1+x}{x} \right) \right) , \tag{3.7}$$

$$g(x) = -\frac{1}{x-1} . \tag{3.8}$$

In this way, fixing M_i and Y_{fi}, all the parameters in the Boltzmann equations are determined, and we can calculate the $B-L$ asymmetry y_Δ just after the second lightest RHN decouples. This asymmetry is washed out when the lightest RHN starts decaying. In this period, the τ and μ Yukawa couplings enter into the equilibrium, and thus the fast interactions distinguish all the three flavors. Therefore, we should divide y_Δ into y_{Δ_τ}, y_{Δ_μ} and y_{Δ_e}, which follow the relation $y_{\Delta_\tau} : y_{\Delta_\mu} : y_{\Delta_e} = \tilde{m}_2^\tau : \tilde{m}_2^\mu : \tilde{m}_2^e$ according to the probabilistic interpretation. This set of asymmetries y_{Δ_f}, $(f = e, \mu, \tau)$ gives the initial condition of the analysis given in the section 2.3.

The neutrino Yukawa couplings Y should be related to the low energy neutrino parameters through the seesaw relation, eq. (2.2). In order to represent the solution of this relation, we adopt the following famous parameterization [20],

$$Y_{fi} = \sum_j \left(U^*\right)_{fj} \sqrt{m_j} R_{ji} \sqrt{M_i}/v. \tag{3.9}$$

Here U is written as the product of a CKM-like mixing matrix V which includes three mixing angles and one CP phase[5] and a phase matrix with two Majorana phases $P = \mathrm{diag}\left(1, \exp\left(i\alpha_{21}/2\right), \exp\left(i\alpha_{31}/2\right)\right)$: $U = VP$ and R is a complex orthogonal matrix which can be decomposed as $R = e^{i\omega_{23}\lambda_7} e^{i\omega_{13}\lambda_5} e^{i\omega_{12}\lambda_2}$ where λ_i are Gell-Mann matrices and ω_{ij} are complex parameters. For simplicity, in this article, we use the following set of the parameters for the light neutrino sector as $m_i = \{0, 9, 50\}$ meV, $\left\{s12^2, s23^2, s13, \delta, \alpha_{21}, \alpha_{31}\right\} = \{0.3, 0.5, 0, 0, 0, 0\}$ for the PMNS matrix, and Majorana masses $M_i = \left\{10^7, 10^{13}, 10^{14}\right\}$ GeV for the RHN.

As representative examples, let us consider the following sets:

$$\begin{aligned}
(\mathrm{I}):&\quad \{\omega_{12}, \omega_{23}, \omega_{13}\} = \{30°, i5°, -1°\} \\
(\mathrm{IIa}):&\quad \{\omega_{12}, \omega_{23}, \omega_{13}\} = \{-88°, (60+i3)°, 3°\} \\
(\mathrm{IIb}):&\quad \{\omega_{12}, \omega_{23}, \omega_{13}\} = \{(-85+i4)°, (50+i20)°, -5.5°\}
\end{aligned} \tag{3.10}$$

For instance, for the example (I), we find

$$\tilde{m} = \begin{pmatrix} 0.68 & 2.04 & 0.02 \\ 0.71 & 2.66 & 25.2 \\ 0.98 & 2.39 & 25.2 \end{pmatrix} \mathrm{meV}, \quad \epsilon = \begin{pmatrix} 10^{-7} & -0.27 & -0.02 \\ 10^{-4} & -95.0 & 11.57 \\ 10^{-4} & 84.3 & -11.28 \end{pmatrix} \times 10^{-6}. \tag{3.11}$$

These show $\{\tilde{m}_1^e, \tilde{m}_1^\mu, \tilde{m}_1^\tau\} = \{0.68, 0.71, 0.98\}\,\mathrm{meV} \lesssim m_*$, and $\{\epsilon_1^e, \epsilon_1^\mu, \epsilon_1^\tau\} = \left\{10^{-13}, 10^{-11}, 10^{-11}\right\}$ are negligibly small. Thus, this is an example of the case (1) in the section 2.3. Using the approximation (3.4), we see the "baryon asymmetry" generated by the N_2 decay is $y_B^0 = 3.36 \times 10^{-10}$. When the temperature decrease to around M_1, N_1 starts decaying, and the asymmetry is washed out in the way investigated in the last section. As mentioned above, in this period, the fast interactions distinguish all the flavor, and the asymmetry should be divided as $\left\{y_{B_e}^0, y_{B_\mu}^0, y_{B_\tau}^0\right\} = \{0.97,\ 1.26,\ 1.13\} \times 10^{-10}$. After the washout, a total asymmetry $y_B = 1.26 \times 10^{-10}$ remains.

In a similar way, (IIa) and (IIb) are examples of the cases (2a) and (2b), respectively. Their results are listed in the table 3.

[5] As a parametrization of V, we adopt the Chau-Keung parametrization (PDG parametrization) [21].

	(I)	(IIa)	(IIb)
$\tilde{m}\,(\mathrm{meV})$	$\begin{pmatrix} 0.68 & 2.04 & 0.02 \\ 0.71 & 2.66 & 25.2 \\ 0.98 & 2.39 & 25.2 \end{pmatrix}$	$\begin{pmatrix} 0.68 & 0.01 & 2.03 \\ 12.0 & 0.01 & 16.3 \\ 27.3 & 0.02 & 1.00 \end{pmatrix}$	$\begin{pmatrix} 1.46 & 10^{-5} & 1.91 \\ 10.43 & 0.421 & 24.26 \\ 27.91 & 0.428 & 6.95 \end{pmatrix}$
$\epsilon\,(10^{-6})$	$\begin{pmatrix} 10^{-7} & -0.27 & -0.02 \\ 10^{-4} & -95.0 & 11.57 \\ 10^{-4} & 84.3 & -11.28 \end{pmatrix}$	$\begin{pmatrix} 10^{-6} & -8.97 & 0.01 \\ 10^{-5} & -31.73 & 0.02 \\ 10^{-5} & -5.02 & -0.02 \end{pmatrix}$	$\begin{pmatrix} 10^{-6} & -0.52 & 10^{-3} \\ 10^{-4} & 341 & -2.70 \\ 10^{-4} & 185 & 0.335 \end{pmatrix}$
$y_B^0\,(10^{-10})$	$\{0.97,\ 1.26,\ 1.13\}$	$\{2.53,\ 2.09,\ 4.28\}$	$-\{0.017,\ 521,\ 530\}$
$y_B\,(10^{-10})$	1.26	1.02	2.00

Table 1: Results of $\tilde{m}$, ϵ, y_B^0 and y_B for the three examples in (3.10).

4. Summary and discussions

In this article, we investigate the washout effect due to the N_1 (inverse) decay, assuming non-vanishing initial lepton asymmetry and negligible lepton asymmetry production in N_1 decay. We show that there is a novel parameter region in addition to those studied in refs. [6, 9]. There, off-diagonal elements of the A-matrix, which are often omitted, play a critical role. This region is where some of $\tilde{m}_1^f$ is comparable to or smaller than m_*, the others are larger than it, and the initial asymmetries on the flavors with small $\tilde{m}_1^f$ are tiny. In this case, if we would omit the off diagonal elements as usual, any initial asymmetries on the flavors with large $\tilde{m}_1^f$ were strongly washed out. In fact, the off diagonal elements transform the asymmetries from those with large $\tilde{m}_1^f$ to those with small ones. Once transformed, such asymmetries are weakly washed out, and thus a sizable total asymmetry may survive.

For completeness, we examine the possibility that the initial asymmetry is generated by the N_2 decay within the thermal leptogenesis scenario. We show a concrete example for each class discussed in the above analysis.

Finally, let us make a comment on an ambiguity of the Boltzmann equations, especially on the factor in front of y_h in eq. (2.10). As briefly discussed below the equation, an approximation is used in the derivation of the Boltzmann equations (2.4), (2.5), and it brings the ambiguity. Because the contribution of this term (y_h) is relatively small, as seen from (2.17) and (2.18), this ambiguity does not affect results so much, $\mathcal{O}\,(10\%)$. In the case (2b), however, the off diagonal element of A-matrix, which is of the same order as the y_h contribution, is critical. In addition, a cancellation occurs between y_h contribution and that of y_l in the off diagonal element when the factor in front of y_h is around 1/4. Thus, the result is largely changed due to the ambiguity. In fact, if we take a factor 1/2 instead of 3/4 as a possible choice, the final baryon asymmetry is reduced to $y_B = 0.83 \times 10^{-10}$ with the same parameters as (IIb) in (3.10). Thus, it is important to make a closer look on the Boltzmann equations before discussing this novel effect quantitatively.

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